15 Limit sets. Lyapunov functions

By this point, considering the solutions to

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in U \subseteq \mathbf{R}^2, \tag{1}$$

I was mostly interested in the behavior of solutions when $t \to \infty$ (sometimes, this is called *asymptotic* behavior of the solutions). It does not mean that the *transient* behavior of the solutions is of no importance, but at this point I would like to concentrate on the case $t \to \infty$. From the previous discussion it should be clear that an equilibrium or a limit cycle can be an ultimate outcome of the system's dynamics. Can we have anything else? How to describe the asymptotic behavior in most general terms? In this lecture I will try to answer these and some other questions by studying the limit sets of the flow of (1). I will also introduce the notion of the Lyapunov function, an extremely powerful device to analyze behavior of (1).

15.1 Limit sets and their properties

Recall that $\boldsymbol{x}(t; \boldsymbol{x}_0)$ denotes the solution to (1) at time t with the initial condition $\boldsymbol{x}(0) = x_0$. Considered as the mapping of U to U parameterized by the time t, it is called *the flow* of system (1). The corresponding geometric object that I study is the orbit $\gamma(\boldsymbol{x}_0)$.

Definition 1. *Point $\bar{\boldsymbol{x}}$ is called an omega limit point of solution $\boldsymbol{x}(t; \boldsymbol{x}_0)$ or orbit $\gamma(\boldsymbol{x}_0)$ if there exists a sequence $t_1, \ldots, t_k, \ldots$ of the time moments such that $t_k \to \infty$ as $k \to \infty$, for which*

$$\boldsymbol{x}(t_k; \boldsymbol{x}_0) \to \bar{\boldsymbol{x}}, \quad k \to \infty$$

holds.

The set of all such points of $\boldsymbol{x}(t; \boldsymbol{x}_0)$ is called the ω-limit (or forward) set of $\boldsymbol{x}(t; \boldsymbol{x}_0)$ or orbit $\gamma(\boldsymbol{x}_0)$ and denoted $\omega(\boldsymbol{x}_0)$.

Similarly, for $t_k \to -\infty$ an alpha limit point and α-limit (or backward) set $\alpha(\boldsymbol{x}_0)$ are defined.

Example 2. If $\hat{\boldsymbol{x}}$ is an equilibrium of (1) then

$$\omega(\hat{\boldsymbol{x}}) = \alpha(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{x}}.$$

Example 3. An asymptotically stable limit cycle is the ω-limit set for at leat some initial conditions (can you think of a sequence of points such that for any point on the limit cycle there is a convergent sequence $\boldsymbol{x}(t; \boldsymbol{x}_0)$?).

By the way, using the limit sets it is possible to define the limit cycle as the closed curve, which is a limit set for some initial conditions that do not belong to the curve itself.

Example 4. Consider

$$\dot{x} = -x, \quad x(t) \in \mathbf{R}.$$

For the equilibrium $\hat{x} = 0$ I have that $\alpha(0) = \omega(0) = 0$. For any other point $x \in \mathbf{R} \setminus \{0\}$ I obviously have

$$\omega(x) = 0, \quad \alpha(x) = \emptyset.$$

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Example 5. Here is a slightly more involved example:

$$\dot{r} = r(a-r),$$
$$\dot{\theta} = \sin^2\theta + (r-a)^2.$$

Here I have three equilibria $\hat{r}_0 = 0$, $(\hat{r}, \hat{\theta}) = (a, 0)$, and $(\hat{r}, \hat{\theta}) = (a, \pi)$. All three of them are unstable. The analysis of the phase portrait shows that the ω-limit set for most of the initial conditions is given by the closed curve composed by two equilibria on the circle, and the heteroclinic trajectories connecting them (recall, that an orbit is called heteroclinic if it connects two equilibria).

Therefore, in the examples I presented it is possible for the limit set of a planar system to be an equilibrium, a limit cycle, or a closed curve that is composed of equilibria and orbits. Can it be anything else?

Limit sets have a lot of nice properties, which I summarize in the following theorem.

Theorem 6. *Limit sets are closed and invariant. If $\boldsymbol{x}(t;\boldsymbol{x}_0)$ is bounded (i.e., $|\boldsymbol{x}(t;\boldsymbol{x}_0)| < M$ for any t for some constant M which may depend on $\boldsymbol{x}_0$), then $\omega(\boldsymbol{x}_0)$ and $\alpha(\boldsymbol{x}_0)$ are non-empty and connected.*

I will leave the proof of this theorem to the reader. Here are a few remarks.

- The limit set is closed means that for any convergent sequence $\bar{\boldsymbol{x}}_k$ belonging to, e.g., $\omega(\boldsymbol{x}_0)$ the limit $\lim_{k\to\infty} \bar{\boldsymbol{x}}_k = \bar{\boldsymbol{x}}$ is also in $\omega(\boldsymbol{x}_0)$.
- The invariance property means that if $\bar{\boldsymbol{x}} \in \omega(\boldsymbol{x}_0)$ then the whole orbit $\boldsymbol{x}(t;\bar{\boldsymbol{x}}) \in \omega(\boldsymbol{x}_0)$.
- The fact that a bounded orbit has a non-empty limit set follows immediately from the fact that any bounded sequence has a convergent subsequence. It also holds that $\omega(\boldsymbol{x}_0)$ is empty if and only if $|\boldsymbol{x}(t;\boldsymbol{x}_0)| \to \infty$.
- Connectedness of $\omega(\boldsymbol{x}_0)$ can be proved by contradiction.

The celebrated Poincaré–Bendixson Theorem classifies the possible limit sets of planar systems.

Theorem 7 (Poincaré–Bendixson)**.** *Consider planar system* (1)*, for which the equilibria are isolated. If the positive orbit $\gamma^+(\boldsymbol{x}_0)$ (i.e., for $t > 0$) is bounded then*

1. *$\omega(\boldsymbol{x}_0)$ is an equilibrium, or*
2. *$\omega(\boldsymbol{x}_0)$ is a closed orbit, or*
3. *For each $\bar{\boldsymbol{x}} \in \omega(\boldsymbol{x}_0)$, $\alpha(\bar{\boldsymbol{x}})$ and $\omega(\bar{\boldsymbol{x}})$ are equilibria.*

15.2 Lyapunov functions and limit sets

In the previous subsection I discussed the limit sets. But how actually can I find them? For this I will need the notion of the *Lyapunov function.* In particular, the following theorem holds.

Theorem 8. *Consider an autonomous system of ODE*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in U \subseteq \mathbf{R}^d,$$

and let $G \subseteq U$. Consider function $V \in \mathcal{C}^{(1)}(G;\mathbf{R})$ and the mapping $t \mapsto V\big(\boldsymbol{x}(t;\boldsymbol{x}_0)\big)$ along the solution $\boldsymbol{x}(t;\boldsymbol{x}_0)$. If the derivative of this mapping $\dot{V}$ is sign definite (i.e., $\dot{V} \leq 0$ or $\dot{V} \geq 0$) then $\omega(\boldsymbol{x}_0) \cap G$ (and $\alpha(\boldsymbol{x}_0) \cap G$) contained in the set $\{\boldsymbol{x} \colon \dot{V}(\boldsymbol{x}) = 0\}$.

Proof. Assume that $\dot{V} \geq 0$ (the other case is treated similarly). Let $\bar{\boldsymbol{x}} \in \omega(\boldsymbol{x}_0)$. This means that there is sequence $(t_k)_{k=1}^{\infty}$ such that $\boldsymbol{x}(t_k; \boldsymbol{x}_0) \to \bar{\boldsymbol{x}}$. This means, by continuity of $\dot{V}$, that $\dot{V}(\bar{\boldsymbol{x}}) \geq 0$. If $\dot{V}(\bar{\boldsymbol{x}}) = 0$ then the theorem is proved. Therefore assume that

$$\dot{V}(\bar{\boldsymbol{x}}) > 0.$$

This implies that $V\big(\boldsymbol{x}(t; \bar{\boldsymbol{x}})\big) > V(\bar{\boldsymbol{x}})$. On the other hand, since V is non-decreasing along the trajectories,

$$V\big(\boldsymbol{x}(t; \boldsymbol{x}_0)\big) \leq V(\bar{\boldsymbol{x}}).$$

Consider sequence $(t+t_k)_{k=1}^{\infty}$ for some fixed t. By the properties of the solutions $\boldsymbol{x}(t+t_k; \boldsymbol{x}_0) = \boldsymbol{x}(t; \bar{\boldsymbol{x}})$, which together with the previous yields a contradiction $V\big(\boldsymbol{x}(t; \bar{\boldsymbol{x}})\big) \leq V(\bar{\boldsymbol{x}})$. Therefore, $\dot{V}(\bar{\boldsymbol{x}}) = 0$. ■

Function V as in the last theorem is called a *Lyapunov function* (but see the next subsection). The derivative $\dot{V}$ is often called the derivative of V *along the vector field* $\boldsymbol{f}$, or *Lie derivative.*

Example 9. Consider again the predator–prey model with intraspecific competition, in dimensionless variables

$$\begin{aligned}\dot{x} &= x(1 - \alpha x - y),\\ \dot{y} &= y(-\gamma - \beta y + x).\end{aligned}$$

I know from the previous analysis that for some parameter values ($\alpha\gamma < 1$) it is possible to have a nontrivial equilibrium $\hat{\boldsymbol{x}}_2 = (\hat{x}, \hat{y})$, which corresponds to the mutual species coexistence. I also found that this equilibrium, when present in $\mathbf{R}^2_+$ is asymptotically stable. In my analysis I was not able to completely discard the possibility of having a closed curve. This example fills this gap.

Consider function, whose form is suggested by the integral of the classical Lotka–Volterra predator–prey model

$$V(x, y) = \hat{x} \log x - x + \hat{y} \log y - y$$

and find its derivative along our vector field:

$$\dot{V} = (\hat{x} - x)(1 - \alpha x - y) + (\hat{y} - y)(-\gamma - \beta y + x) = \alpha(\hat{x} - x)^2 + \beta(\hat{y} - y)^2.$$

For any $(x, y) \in \mathbf{R}^2_+$ $\dot{V}$ is non-negative and is equal to zero only at $(\hat{x}, \hat{y})$. Therefore, this point is the only candidate for the ω-limit set, and therefore it is globally asymptotically stable for the system.

Sometimes the set of zeros of $\dot{V}$ is large, and I would like to identify those points which are actually in the limit sets. This can be done with

Theorem 10 (LaSalle)**.** *Let $\hat{G} = \{\boldsymbol{x} \in G \colon \dot{V}(\boldsymbol{x}) = 0\}$ be the set of zeros of the derivative of a Lyapunov function along given vector field $\boldsymbol{f}$, and let G be positive invariant. Then the orbits of the vector field tend to the maximal invariant subset of $\hat{G}$.*

15.3 Lyapunov functions and stability of equilibria

In the previous subsection a function V was called a Lyapunov function for vector field $\boldsymbol{f}$ in the domain G if it is sign definite in G along the orbits of $\boldsymbol{f}$. A canonical, and slightly more stringent, definition of Lyapunov function is as follows.

Definition 11. *Let $\hat{\boldsymbol{x}}$ be an equilibrium of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x}(t) \in U \subseteq \mathbf{R}^d$. A function $V \in \mathcal{C}^{(1)}(G; \mathbf{R})$, where G is a neighborhood of $\hat{\boldsymbol{x}}$, is called a Lyapunov function for $\hat{\boldsymbol{x}}$ in G if:*

- *$V(\hat{\boldsymbol{x}}) = 0$ and $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in G \setminus \{\hat{\boldsymbol{x}}\}$.*
- *$\dot{V}(\boldsymbol{x}) \leq 0$ for all $\boldsymbol{x} \in G$.*

If instead of the second condition, the condition

- *$\dot{V}(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \in G \setminus \{\hat{\boldsymbol{x}}\}$*

holds, then V is called a strict Lyapunov function.

Using this definition, it is possible to prove

Theorem 12 (Lyapunov)**.** *Let $\hat{\boldsymbol{x}}$ be an equilibrium of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x}(t) \in U \subseteq \mathbf{R}^d$. If there exists a Lyapunov function for $\hat{\boldsymbol{x}}$ then $\hat{\boldsymbol{x}}$ is stable, if there exists a strict Lyapunov function for $\hat{\boldsymbol{x}}$ than $\hat{\boldsymbol{x}}$ is asymptotically stable.*

There are no universal methods to construct Lyapunov functions, though.

Example 13. Consider first the classical Lotka–Volterra model

$$\begin{aligned} \dot{N} &= aN - bPN, \\ \dot{P} &= cPN - dP. \end{aligned}$$

Recall that I was able to integrate this system, and found that the solutions are the level sets of

$$H(N, P) = bP + cN - a \log P - d \log N.$$

If one considers a function

$$V(N, P) = H(N, P) - H(\hat{N}, \hat{P}), \quad \hat{N} = \frac{d}{c}, \hat{P} = \frac{a}{b}$$

then it can be checked that V is a Lyapunov function for $(\hat{N}, \hat{P})$. Hence the equilibrium, as we already saw, Lyapunov stable. Here, actually, $\dot{V} \equiv 0$.

Consider again the predator–prey model with intraspecific competition, in dimensionless variables

$$\begin{aligned} \dot{x} &= x(1 - \alpha x - y), \\ \dot{y} &= y(-\gamma - \beta y + x). \end{aligned}$$

And let

$$H(x, y) = \hat{x} \log x - x + \hat{y} \log y - y.$$

One can check that

$$V(x, y) = H(\hat{x}, \hat{y}) - H(x, y)$$

is a strict Lyapunov function for $(\hat{x}, \hat{y})$.

It is important to mention that Lyapunov functions allow to infer global aspects of the behavior of the orbits. For example, the following is true.

Consider again the same system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x}(t) \in U \subseteq \mathbf{R}^d$ with an asymptotically stable equilibrium $\hat{\boldsymbol{x}}$. By definition, the *basin of attraction* $B(\hat{\boldsymbol{x}})$ of $\hat{\boldsymbol{x}}$ is the set of initial conditions such that $\boldsymbol{x}(t; \boldsymbol{x}_0) \to \hat{\boldsymbol{x}}$ for $t \to \infty$. It can be shown that if V is a strict Lyapunov function for $\hat{\boldsymbol{x}}$, then the sets $G_c = \{\boldsymbol{x} \in G \colon V(\boldsymbol{x}) \leq c\}$ are in the basin of attraction of $\hat{\boldsymbol{x}}$. In the last example, the whole $\mathbf{R}^2_+$ is in the basin of attraction of $(\hat{x}, \hat{y})$.